Quantum Cosmology on Ultrametric and Noncommutative Spaces

Ljubisa Nesic,
Department of Physics, Faculty of Sciences and Mathematics,
University of Nis, Serbia and Montenegro
E-mail: nesiclj@junis.ni.ac.yu

in collaboration with D. Dimitrijevic and G. Djordjevic.

- Motivation.
- Mathematics.
- Quantum Mechanics on nonarchimedean/p-adic/ spaces.
- Minisuperspace Quantum Cosmology as a Quantum Mechanics over minisuperspace.
- Kaluza-Klein (4+1) dimensional empty model.

MOTIVATION

- Appearance of the Nature at the small distances.
- Archimedean or nonarchimedean?

Concept of distance.

- We measure distances (between real and rational numbers) using the absolute value on the real numbers, (natural metric on the field of the real numbers *R*).
- How did the real numbers enter analysis?
- The field of real numbers **R** is the result of *completing* the field of rationals **Q** with the respect to the usual absolute value **|.|**.
- The field **Q** is Causchi incomplete with respect to the usual absolute value **|.|**

{1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, …} =>SQRT(2)
SQRT(2) notin Q!!!

(A sequence $\{x_n\}$ is called a Caushy sequence with the respect to the norm |.| if it satisfies the following property: Given any a>0, there exists some N such that $m,n > N$ implies $| x_m - x_n | < a$. Basically, a sequence is Caushy if its terms became “arbitrarily close” with respect to the norm |.|)

- A map **|.|** from the rationals to the non-negative reals is called a *norm* (or valuations) if it satisfies the three following conditions:

(1) $|x|=0$ if and only if $x=0$
(2) $\forall x,y$ we have $|xy|=|x||y|$ (norm of product is product of norms)
(3) $\forall x,y$ we have $|x+y|\leq|x|+|y|$ (the triangle inequality)

- The usual absolute value **|.|** clearly satisfies these properties, but what other kinds of norms can exist?

- There is *trivial norm* |*x*|=1 for all rationals *x* except 0, with |0|=0.

- ***Besides the usual completion of field of rationals (which leads to the R) there is a non-obvious completion of rationals . These are the p-adic fields Q_p where p is some fixed prime number (Kurt Hensel in 1902),* (*p*=2,3,5,7,...).**

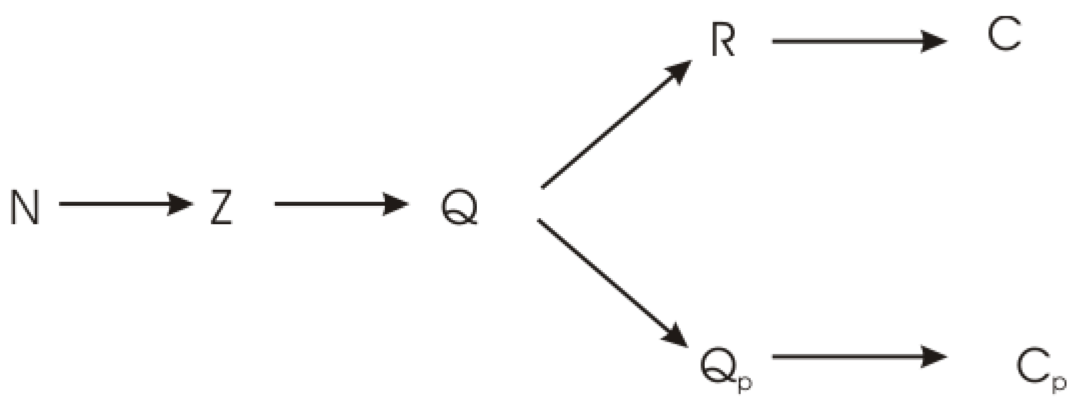
- Each p-adic field $\mathbf{Q}_p$ is defined by completing **Q** with respect to the "new" norm $|.|_p$ which is defined as follows:

- $0 \neq x \in Q$, $x = p^{\gamma} \frac{m}{n}$ where *m* and *n* are nonzero integers, neither divisible by the prime *p*, and γ is an integer

$$|x|_p = p^{-\gamma}.$$

- If we further define $|0|_p=0$, then that $|.|_p$ satisfies the necessary conditions above to be a norm on **Q**.

- The set of equivalence classes of Caushy (with respect to $|.|_p$) sequences has a natural field structure. It is a completion of **Q** which we call the field of ***p*-adic numbers**



Nonarchimedean norm

- There is a stronger inequality for an absolute value **|.|** than triangle inequality which is known as the **ultrametric** inequality or strong triangle inequality:

$$|x+y| \leq \max\{|x|,|y|\}$$

- Any norm **|.|** satisfying this is called **nonarchimedean** or **ultrametric**.
- The usual norm on the real line is clearly archimedean – in fact there is an archimedean axiom:

"Any given large segment of a straight line can be surpassed by successive addition of small segments along the same line."

- A more formal statement of the axiom would be that if $0<|x|<|y|$ then there is some positive integer n such that $|nx|>|y|$.

Can we measure distances as small as we want?

- It is very difficult to imagine a situation where this axiom does not hold, but the archimedean axiom breaks down at the Planck scale, (10^{-33} m, 10^{-44} s).
- Below this scale, distances and durations cannot scaled up in order to produce macroscopic distances and durations.
- In other words, we must abandon the archimedean axiom at very small distances.
- How can one construct a physical theory corresponding to a non-archimedean geometry?

geometry ←→ number system

- What should be used instead of real numbers?
- In computations in everyday life, in scientific experiments and on computers we are dealing with integers and fractions, that is with rational numbers and we newer have dealings with irrational numbers.
- Results of any practical action we can express only in terms of rational numbers which are considered to have been given to us by God. What norms do exist on **Q**?
- There is a remarkable Ostrowski theorem describing all norms on **Q**. According to this theorem: *any nontrivial norm on **Q** is equivalent to either ordinary absolute value or p-adic norm for some fixed prime number p.*
- *p*-adic fields are **nonarchimedean**, it is natural to consider formulations of physics in terms of $\mathbf{Q}_p$ rather than **R**
- "Which *p* do we use?"
- The adeles constitute a locally compact topological ring $\mathbf{A_Q}$, individually taking the form

$$(a_\infty; a_2, a_3, a_5, \cdots)$$

In all but a finite number of cases a_p is a p-adic integer ($|a_p|_p \leq 1$).

p-ADIC FUNCTIONS AND INTEGRATION

- There are primary two kinds of analyses on $\mathbf{Q}_p$: $\mathbf{Q}_p \to \mathbf{Q}_p$ (class.) and $\mathbf{Q}_p \to \mathbf{C}$ (quant.).
- Usual complex valued functions of p-adic variable, which are employed in mathematical physics, are :

(i) an additive character $\chi_p(x) = \exp 2\pi i \{x\}_p$, where $\{x\}_p$ is the fractional part of $x \in Q_p$,

(ii) multiplicative character $\pi_s(x) = |x|_p^s$, where $s \in C$, and

(iii) locally constant functions with compact support, like $\Omega(|x|_p)$, where

$$\Omega(|x|_p) = \begin{cases} 1 & |x|_p \leq 1 \\ 0 & |x|_p > 1 \end{cases}.$$

- There is well defined Haar measure and integration. Important integrals are

$$\int_{Q_p} \chi_p(ayx)dx = \delta_p(ay) = |a|_p^{-1} \delta_p(y),\ a \neq 0$$

$$\int_{Q_p} \chi_p(\alpha x^2 + \beta x)dx = \lambda_p(\alpha)\,|2\alpha|_p^{-1/2}\, \chi_p\left(-\frac{\beta^2}{4\alpha}\right),\ \alpha \neq 0\,,$$

- The number theoretic function $\lambda_p(x)$ is a map $\lambda_p : Q_p^* \to C$ defined as follows:

$$\lambda_p(x) = \begin{cases} 1, & m = 2j \quad & p \neq 2 \\ \left(\frac{x_0}{p}\right), & m = 2j+1 & p \equiv 1 (\mathrm{mod}\, 4) \\ i\left(\frac{x_0}{p}\right), & m = 2j+1 & p \equiv 3 (\mathrm{mod}\, 4) \end{cases},$$

$$\lambda_2(x) = \begin{cases} \frac{1}{\sqrt{2}}[1+(-1)^{x_1} i] & m = 2j \\ \frac{1}{\sqrt{2}}(-1)^{x_1+x_2}[1+(-1)^{x_1} i] & m = 2j+1 \end{cases},$$

$x = p^m(x_0 + x_1 p + x_2 p^2 + \cdots)$, and $m, j \in Z$.

- $(\frac{x_0}{p})$ is the Legandre symbol defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & if \quad a \equiv y^2 (\text{mod } p) \\ -1 & if \quad a \neq y^2 (\text{mod } p) \\ 0 & if \quad a \equiv 0 (\text{mod } p) \end{cases}.$$

Properties of λ_p-function

$$\lambda_p(a^2 x) = \lambda_p(x), \quad \lambda_p(x)\lambda_p(-x) = 1, \quad \lambda_p(x)\lambda_p(y) = \lambda_p(x+y)\lambda_p(x^{-1}+y^{-1})$$

$$|\lambda_p(x)|_\infty = 1, \quad a \neq 0.$$

Real analogues of integrals

$$\int_{Q_\infty} \chi_\infty(ayx)dx = \delta_\infty(ay) = |a|_\infty^{-1} \delta_\infty(y), a \neq 0$$

$$\int_{Q_\infty} \chi_\infty(\alpha x^2 + \beta x)dx = \lambda_\infty(\alpha)|2\alpha|_\infty^{-1/2} \chi_\infty\left(-\frac{\beta^2}{4\alpha}\right), \alpha \neq 0,$$

$$Q_\infty \equiv R, \quad \chi_\infty(x) = \exp(-2\pi i x)$$

$$\lambda_\infty(x) = \sqrt{\frac{sign\, x}{i}}$$

QUANTUM MECHANICS ON ADELIC SPACES

Reasons to use p-adic numbers and adeles in quantum physics:

- the field of rational numbers **Q**, which contains all observational and experimental numerical data, is a dense subfield not only in **R** but also in the fields of p-adic numbers $\mathbf{Q}_p$
- there is an analysis within and over $\mathbf{Q}_p$ like that one related to **R**
- general mathematical methods and fundamental physical laws should be invariant [I.V. Volovich, *Number theory as the ultimate physical theory*, CERN preprint, CERN-TH.4781/87 (July 1987)] under an interchange of the number fields **R** and $\mathbf{Q}_p$
- there is a quantum gravity uncertainty ($\Delta x \geq l_0 = \sqrt{\hbar G / c^3}$), when measures distances around the Planck length l_0, which restricts priority of Archimedean geometry based on the real numbers and gives rise to employment of non-Archimedean geometry related to p-adic numbers
- it seems to be quite reasonable to extend standard Feynman's path integral method to non-Archimedean spaces,
- adelic quantum mechanics [B. Dragovich, Adelic Model of Harmonic Oscillator, *Theor,. Math. Phys,* **101** (1994) 1404-1412] is consistent with all.

Adelic quantum mechanics

$$(L_2(A), W(z), U(t))$$

- $L_2(A)$ - adelic Hilbert space,
- $W(z)$ - Weyl quantization of complex-valued functions on adelic classical phase space,
- $U(t)$ - unitary representation of an adelic evolution operator on $L_2(A)$

Dynamics of p-adic quantum model?

$$U_p(t)\psi^{(p)}(x)=\int_{Q_p} K_t^{(p)}(x,y)\psi^{(p)}(y)dy$$

p-adic quantum mechanics is given by a triple

$$(L_2(Q_p),W_p(z_p),U_p(t_p))$$

Adelic evolution operator $U(t)$ is defined by

$$U(t)\psi(x)=\int_A K_t\,(x,y)\psi(y)dy=\prod_{v=\infty,2,3,\dots,p,\dots}\int_{Q_v} K_t^v(x_v,y_v)\psi^{(v)}(y_v)dy_v$$

The eigenvalue problem for $U(t)$ reads

$$U(t)\psi_\alpha(x)=\chi(E_\alpha t)\psi_\alpha(x)$$

The main problem in our approach is computation of p-adic propagator $K_p(x'',t'';x',t')$ in Feynman's path integral method, i.e.

$$K_p(x'',t'';x',t')=\int_{(x',t')}^{(x'',t'')}\chi_p(-\tfrac{1}{h}\int_{t'}^{t''}L(\dot q,q,t))Dq$$

Exact general expression for propagator

$$K_p(x'',t'';x',t')=\lambda_p(-\tfrac{1}{2h}\tfrac{\partial^2\bar S}{\partial x'\partial x''})\times\left|\tfrac{1}{h}\tfrac{\partial^2\bar S}{\partial x'\partial x''}\right|_p^{1/2}\chi_p\left(-\bar S(x'',t'';x',t')\right)$$

$$K_p(x'',y'',z'',t'';x',y',z',t')=\lambda_p\left(\det\begin{pmatrix}-\frac{1}{2}\frac{\partial^2\bar S}{\partial x'\partial x''} & -\frac{1}{2}\frac{\partial^2\bar S}{\partial x'\partial y''} & -\frac{1}{2}\frac{\partial^2\bar S}{\partial x'\partial z''}\\ -\frac{1}{2}\frac{\partial^2\bar S}{\partial y'\partial x''} & -\frac{1}{2}\frac{\partial^2\bar S}{\partial y'\partial y''} & -\frac{1}{2}\frac{\partial^2\bar S}{\partial y'\partial z''}\\ -\frac{1}{2}\frac{\partial^2\bar S}{\partial z'\partial x''} & -\frac{1}{2}\frac{\partial^2\bar S}{\partial z'\partial y''} & -\frac{1}{2}\frac{\partial^2\bar S}{\partial z'\partial z''}\end{pmatrix}\right)$$

$$\times\left|\det\begin{pmatrix}-\frac{\partial^2\bar S}{\partial x'\partial x''} & -\frac{\partial^2\bar S}{\partial x'\partial y''} & -\frac{\partial^2\bar S}{\partial x'\partial z''}\\ -\frac{\partial^2\bar S}{\partial y'\partial x''} & -\frac{\partial^2\bar S}{\partial y'\partial y''} & -\frac{\partial^2\bar S}{\partial y'\partial z''}\\ -\frac{\partial^2\bar S}{\partial z'\partial x''} & -\frac{\partial^2\bar S}{\partial z'\partial y''} & -\frac{\partial^2\bar S}{\partial z'\partial z''}\end{pmatrix}\right|_p^{1/2}\chi_p\left(-\bar S(x'',y'',z'',t'';x',y',z't')\right)$$

where $\bar S(x'',y'',z'',t'';x',y',z',t')$ is the action for classical trajectory.

Illustration of exactly soluble p-adic and adelic quantum mechanical models,

- a free particle and harmonic oscillator [VVZ, Dragovich]
- a particle in a constant field, [Djordjevic, Dragovich]
- a free relativistic particle[Djordjevic, Dragovich, Nesic]
- a harmonic oscillator with time-dependent frequency [Djordjevic, Dragovich]

Resume of AQM: AQM takes in account ordinary as well as p-adic effects and may me regarded as a starting point for construction of more complete quantum cosmology and string/M theory. In the low energy limit AQM effectively becomes the ordinary one.

The form of adelic wave function

$$\psi = \psi_\infty(x_\infty)\cdot\prod_{p\in M}\psi_p(x_p)\cdot\prod_{p\notin M}\Omega(|x|_p)$$

ADELIC QUANTUM COSMOLOGY

- The main task of AQC is to describe the very early stage in the evolution of the Universe.
- At this stage, the Universe was in a quantum state, which should be described by a wave function (complex valued and depends on some real parameters).
- But, QC is related to Planck scale phenomena - it is natural to reconsider its foundations.
- We maintain here the standard point of view that the wave function takes complex values, but we treat its arguments in a more complete way!
- We regard space-time coordinates, gravitational and matter fields to be adelic, i.e. they have real as well as p-adic properties simultaneously.
- There is no Schroedinger and Wheeler-De Witt equation for cosmological models.
- **Feynman's path integral method was exploited [Drag. Nes.] and minisuperspace cosmological models are investigated as a model of adelic quantum mechanics [Drag. Djordj. Nes. Vol.].**
- Adelic minisuperspace quantum cosmology is an application of adelic quantum mechanics to the cosmological models.
- Path integral approach to standard quantum cosmology, the starting point is Feynman's path integral method

$$< h_{ij}^{''}, \phi^{''}, \Sigma^{''} \mid h_{ij}^{'}, \phi^{'}, \Sigma^{'} >_{\infty} = \int D(g_{\mu\nu})_{\infty} D(\phi)_{\infty} \chi_{\infty}(-S_{\infty}[g_{\mu\nu}, \phi])$$

p-adic complex valued cosmological amplitude

$$< h_{ij}^{''}, \phi^{''}, \Sigma^{''} \mid h_{ij}^{'}, \phi^{'}, \Sigma^{'} >_{p} = \int D(g_{\mu\nu})_{p} D(\phi)_{p} \chi_{p}(-S_{p}[g_{\mu\nu}, \phi])$$

Kaluza-Klein (4+1) - dimensional "empty" model

- FRW metrics (F. Darabi, *Phys.Lett.* **B615** (2005) 141-145)

$$ds^2 = -N^2 dt^2 + R^2(t)\frac{dr_i dr^i}{\left(1+\frac{kr^2}{4}\right)^2} + a^2(t)d\rho^2$$

  - Einstein – Hilbert action for empty model

$$S = \int \sqrt{-g}\,(\tilde{R} - \Lambda)\,dt\,d^3r\,d\rho$$

  - Lagrangian

$$L = \frac{1}{2N}Ra\dot{R}^2 + \frac{1}{2N}R^2\dot{R}\dot{a} - \frac{1}{2}kNRa + \frac{1}{6}N\Lambda R^3 a$$

Commutative model over real space

- By defining $\omega^2 = -\frac{2\Lambda}{3}\ (\Lambda < 0)$
- Changing $u = \frac{1}{\sqrt{8}}[R^2 + Ra - \frac{3k}{\Lambda}],\ v = \frac{1}{\sqrt{8}}[R^2 - Ra - \frac{3k}{\Lambda}]$
- Lagrangian (oscillator-ghost-oscillator system)

$$L = \frac{1}{2N}(\dot{u}^2 - N^2\omega^2 u^2) - \frac{1}{2N}(\dot{v}^2 - N^2\omega^2 v^2)$$

- Classical action

$$\bar{S}(u'',v'',N;u',v',0) = \frac{1}{2}\omega[(u''^2 + u'^2 - v''^2 - v'^2)\cot N\omega + (v'v'' - u'u''\frac{2}{\sin N\omega})]$$

$$K(u'',v'',N;u',v',0) = \frac{\omega}{\sin N\omega}\exp[2\pi i\bar{S}(u'',v'',N;u',v',0)]$$

- Wheeler-DeWitt equation

$$\left(\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}-\omega^2 u^2+\omega^2 v^2\right)\Psi(u,v)=0$$

- Oscillator-ghost-oscillator solutions belonging to the Hilbert space $H^{(m_1,m_2)}(L^2)$ as

$$\Psi^{(m_1,m_2)}(u,v)=\sum_{l=0}^{\infty}c_l\Phi_l^{(m_1,m_2)}(u,v)$$

with $m_1,m_2\geq 0$ and $c_l\in C$. The basis solutions $\Phi_l^{(m_1,m_2)}(u,v)$ are separable as

$$\Phi_l^{(m_1,m_2)}(u,v)=\alpha_{m_2+(2m_2+1)l}()u\beta_{m_1+(2m_1+1)l}(v)$$

with normalized solutions

$$\alpha_n(u)=\left(\frac{\omega}{\pi}\right)^{1/4}\frac{e^{-\omega u^2/2}}{\sqrt{2^n n!}}H_n(u\sqrt{\omega})$$

$$\beta_n(v)=\left(\frac{\omega}{\pi}\right)^{1/4}\frac{e^{-\omega v^2/2}}{\sqrt{2^n n!}}H_n(v\sqrt{\omega})$$

Commutative Kaluza-Klein (4+1) - dimensional "empty" model over p-adic space

$$ds^2 = -N^2 dt^2 + R^2(t)\frac{dr_i dr^i}{\left(1+\frac{kr^2}{4}\right)^2} + a^2(t)d\rho^2$$

$$S = \int \sqrt{-g}\,(\tilde{R} - \Lambda)\,dt\,d^3r\,d\rho$$

- p-Adic Lagrangian in the minisuperspace (R, a)

$$L = \frac{1}{2N}Ra\dot{R}^2 + \frac{1}{2N}R^2\dot{R}\dot{a} - \frac{1}{2}kNRa + \frac{1}{6}N\Lambda R^3 a$$

- P-adic Classical action

$$\bar{S}_p(u'', v'', N; u', v', 0)$$
$$= \frac{1}{2}\omega[(u''^2 + u'^2 - v''^2 - v'^2)\cot N\omega + (v'v'' - u'u''\frac{2}{\sin N\omega})]$$

- The propagator is

$$K_p(u'', v'', N; u', v', 0)$$
$$= \frac{1}{|N|_p}\chi_p\left(\frac{\omega(u''^2 + u'^2 - v'^2 - v''^2)}{2\tan N\omega} + \frac{\omega(v'v'' - u'u'')}{\sin N\omega}\right)$$

- In the region of convergence $G_p = \{x \in Q_p : |x|_p \le |2p|_p\}$
- $\Psi_p(u, v) = \Omega(|u|_p)\Omega(|v|_p)$
- $\Psi_p(u, v) = \Omega(p^\nu |u|_p)\Omega(p^\mu |v|_p),\ \nu, \mu = 1,2,3,\ldots$

$$\Psi_p(u,v) = \begin{cases} \delta(p^\nu - |u|_p)\delta(p^\mu - |v|_p) & |N|_p \le p^{2\nu-2}, |N|_p \le p^{2\mu-2} \\ \delta(2^\nu - |u|_2)\delta(2^\mu - |v|_2) & |N|_2 \le 2^{2\nu-3}, |N|_2 \le 2^{2\mu-3} \end{cases},$$

Noncommutative case

- In the previous section we assume that in (u,v) minisuperspace holds

$$[u,v]=0,\quad [u,p_u]=[v,p_v]=i\hbar,\quad [p_u,p_v]=0,$$

$$p_u=\frac{\dot u}{N},\ p_v=\frac{\dot v}{N}$$

- Noncommutative case - the same Lagrangian, a new algebra

$$[u,v]=i\theta,\quad [u,p_u]=[v,p_v]=i\hbar,\quad [p_u,p_v]=0.$$

- Transformation

$$u=u-\frac{\theta}{2}p_v,\ v=v+\frac{\theta}{2}p_u$$

- Model as commuting one with Lagrangian

$$L_\theta=\frac{\omega^2}{\omega_\theta^2}\left[\frac{1}{2N}(\dot u^2-N^2\omega_\theta^2u^2)-\frac{1}{2N}(\dot v^2-N^2\omega_\theta^2v^2)+\frac{1}{2N}\omega_\theta^2\theta(\dot uv+u\dot v)\right]$$

$$\omega_\theta^2=\frac{\omega^2}{1+\frac{\omega^2\theta^2}{4}}$$

- Equations of motion

$$\ddot u+N^2\omega_\theta^2u=0,\quad \ddot v+N^2\omega_\theta^2v=0$$

- Solutions

$$u(t)=A\cos N\omega_\theta t+B\sin N\omega_\theta t,\quad v(t)=C\cos N\omega_\theta t+D\sin N\omega_\theta t$$

- Classical action

$$\bar{S}_\theta(u'',v'',N;u',v',0)=\frac{1}{2}\omega\sqrt{1+\frac{\omega^2\theta^2}{4}}$$
$$\left[(u''^2+u'^2-v''^2-v'^2)\cot N\omega_\theta-(u'u''-v'v'')\frac{2}{\sin N\omega_\theta}+\frac{\theta\omega_\theta}{N}(u''v''-u'v')\right]$$

- Quantum propagator

$$K_\theta(u'',v'',N;u',v',0)=\sqrt{1+\frac{\omega^2\theta^2}{4}}\sqrt{\frac{\omega^2}{\sin^2 N\omega_\theta}}\exp(2\pi i\bar{S}_\theta(u'',v'',N;u',v',0))$$

- Commutative regime - $\theta=0$

$$K_0(u'',v'',N;u',v',0)=\frac{\omega}{\sin N\omega}\chi_\infty(-\bar{S}(u'',v'',N;u',v',0))$$

Discretization of minisuperspace coordinates

It is shown that there is possibility of construction adequate p-adic (4+1)-dim empty Kaluza-Klein model.

It is shown that there exist adelic ground states of the form

$$\psi(u,v)=\psi_\infty^\pm(u_\infty,v_\infty)\prod_{p\in M}\psi_p(u_p,v_p)\prod_{p\notin M}\Omega(|u_p|_p)\Omega(|v_p|_p)$$

- Adopting the usual probability interpretation of the wave function, we have

$$|\psi(u,v)|_\infty^2=|\psi_\infty^\pm(u_\infty,v_\infty)|_\infty^2\prod_{p\in M}|\psi_p(u_p,v_p)|_\infty^2\prod_{p\notin M}\Omega(|u_p|_p)\Omega(|v_p|_p)$$
$$(\Omega(|u|_p))^2=\Omega(|u|_p)$$

- At the rational points *u* and *v* and for $M=\varnothing$

$$|\psi(u,v)|_\infty^2=\begin{cases}|\psi_\infty^\pm(u,v)|_\infty^2 & u,v\in Z\\ 0 & u,v\in Q\setminus Z\end{cases}$$

(4+D)-DIMENSIONAL COSMOLOGICAL MODELS OVER THE FIELD OF REAL NUMBERS

4+*D* – dimensional Kaluza-Klein cosmology with RW type metric

$$ds^2 = -\tilde{N}^2 dt^2 + R^2(t)\frac{dr_i dr^i}{\left(1+\frac{kr^2}{4}\right)^2} + a^2(t)\frac{d\rho_a d\rho^a}{\left(1+k'\rho^2\right)^2}$$

This model is describing an accelerating universe with dynamical compactification of extra dimensions.

- The form of the energy-momentum tensor is

$$T_{AB} = diag(-\rho, p, p, p, p_D, p_D, ..., p_D),$$

If we want the matter is to be confined to the four-dimensional universe, we set all $p_D = 0$.

Energy-momentum tensor

$$p_\chi = \left(\frac{m}{3} - 1\right)\rho_\chi$$

Dimensionally extended Einstein-Hilbert action

$$S = \int \sqrt{-g}\ \tilde{R}\, dt\, d^3 r\, d^D \rho + S_m = \kappa \int dt L$$

$$L = \frac{1}{2\tilde{N}} R a^D \dot{R}^2 + \frac{D(D-1)}{12\tilde{N}} R^3 a^{D-2}\dot{a}^2 + \frac{D}{2\tilde{N}} R^2 a^{D-1}\dot{R}\dot{a} - \frac{1}{2}k\tilde{N}Ra^D + \frac{1}{6}\tilde{N}\rho_\chi R^3 a^D$$

Closed universe ($k = 1$), continuity equation

$$\dot{\rho}_\chi R + 3(p_\chi + \rho_\chi)\dot{R} = 0$$

energy density

$$\rho_\chi(R)=\rho_\chi(R_0)\left(\frac{R_0}{R}\right)^m .$$

If we define $\Lambda \equiv \rho_\chi(R)$

$$L=\frac{1}{2\tilde{N}}Ra^D\dot{R}^2+\frac{D(D-1)}{12\tilde{N}}R^3a^{D-2}\dot{a}^2+\frac{D}{2\tilde{N}}R^2a^{D-1}\dot{R}\dot{a}$$
$$-\frac{1}{2}\tilde{N}Ra^D+\frac{1}{6}\tilde{N}\Lambda R^3a^D$$

Growth of the scaling factor R, leads to the decrease of the cosmological constant

$$\Lambda(R)=\Lambda(R_0)\left(\frac{R_0}{R}\right)^m .$$

If we take $m=2$, $\Lambda(R_0)R_0^2=3$, $\tilde{N}(t)=R^3(t)a^D(t)N$,

$$L=\frac{1}{2N}\frac{\dot{R}^2}{R^2}+\frac{D(D-1)}{12N}\frac{\dot{a}^2}{a^2}+\frac{D}{2N}\frac{\dot{R}\dot{a}}{Ra}$$

Classical solutions

$$R(t)=Ae^{\alpha t},\quad a(t)=Be^{\beta t}$$

For $R(0)=a(0)=l_P$

$$R(t)=l_Pe^{\alpha t},\quad a(t)=l_Pe^{\beta t}$$

For $D=1$

$$R(t)=l_Pe^{Ht},\quad a(t)=l_Pe^{-Ht},\quad H=\dot{R}/R-Hubble\quad parameter$$

accelerating universe, contracting internal space (with same rates).

For $D>1$

$$R(t) = l_P e^{Ht}, \quad a(t)_\pm = l_P e^{\frac{2Ht}{D}[-1 \pm \sqrt{1-2/3(1-1/D)}]^{-1}}$$

$$R(t)_\pm = l_P e^{\frac{D\beta t}{2}[-1 \pm \sqrt{1-2/3(1-1/D)}]}, \quad a(t) = l_P e^{\beta t}.$$

Quantum solutions

Wheeler-DeWitt equation

$$H\psi(R,a) = 0$$

$X = \ln R$, $Y = \ln a$

$$\left[(D-1)\frac{\partial^2}{\partial X^2} + \frac{6}{D}\frac{\partial^2}{\partial Y^2} - 6\frac{\partial}{\partial X}\frac{\partial}{\partial Y}\right]\psi(X,Y) = 0$$

$x = X\dfrac{3}{D+3} + Y\dfrac{D}{D+3}$, $y = \dfrac{X-Y}{D+3}$

$$\left\{-3\frac{\partial^2}{\partial x^2} + \frac{D+2}{D}\frac{\partial^2}{\partial y^2}\right\}\psi(x,y) = 0$$

with four possible solutions

$$\psi_D^\pm(x,y) = A^\pm e^{\pm\sqrt{\gamma/3}x \pm \sqrt{\gamma D/(D+2)}y},$$

$$\psi_D^\pm(x,y) = B^\pm e^{\pm\sqrt{\gamma/3}x \mp \sqrt{\gamma D/(D+2)}y}.$$

(4+D)-DIMENSIONAL MODEL OVER THE FIELD OF P-ADIC NUMBERS

Lagrangian of the model

$$L = \frac{1}{2N}\dot{X}^2 + \frac{D(D-1)}{12N}\dot{Y}^2 + \frac{D}{2N}\dot{X}\,\dot{Y}\,.$$

Solutions

$$X = C_1 t + C_2,\ Y(t) = C_3\,t + C_4.$$

Classical action

$$\bar{S}(X'',Y'',N;X',Y',0)$$
$$= \frac{1}{2N}(X''-X')^2 + \frac{D(D-1)}{12N}(Y''-Y')^2 + \frac{D}{2N}(X''-X')(Y''-Y')\,.$$

Kernel of p-adic operator of evolution

$$K_p(X'',Y'',N;X',Y',0)$$
$$= \lambda_p\left(\frac{D(D+2)}{48N^2}\right)\left|\frac{D(D+2)}{12N^2}\right|_p^{1/2}\chi_p(-\bar{S}(X'',Y'',N;X',Y',0))$$

change $x = X\frac{3}{D+3} + Y\frac{D}{D+3},\ y = \frac{X-Y}{D+3}$

$$\bar{S}(x'',y'',N;x',y',0)$$
$$= \frac{1}{2N}\left(1+\frac{D(D+5)}{6}\right)(x''-x')^2 - \frac{1}{2N}D(D+3)(y''-y')^2,$$

$$K_p(x'',y'',N;x',y',0) = \lambda_p\left(\frac{6+D(D+5)}{12N}\right)\lambda_p\left(-\frac{D(D+3)}{2N}\right)$$
$$\times\left|\frac{D(D+3)}{2N^2}\left(1+\frac{D(D+5)}{6}\right)\right|_p^{1/2}\chi_p(-\bar{S}).$$

p-adic ground state wave function in (x,y) minisuperspace

$$\psi_p(x,y) = \Omega(|x|_p)\Omega(|y|_p).$$

Conditions

$$|N|_p \leq |1 + D(D+5)/6|_p$$

$$|N|_p \leq |D(D+3)|_p,\ p \neq 2.$$

p-adic ground state wave function in (X,Y) minisuperspace

$$\psi_p(X,Y) = \Omega\left(\left|(1 - \frac{D}{D+3})X + \frac{D}{D+3}Y\right|_p\right)\Omega\left(\left|\frac{X-Y}{D-3}\right|_p\right)$$