A Modification of the Analytic Continuation by Duality Method

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Abstract

The Analytic Continuation by Duality (ACD) method has found applications in low-energy QCD phenomenology as well as in technicolor models. The original ACD method approximates $1/s$ kernel in a dispersive integral by polynomials whose coefficients are found by minimization of an L_p norm. The method is unstable with oscillatory behavior of the estimates, due in part to the nature of the best-L_p approximations. A recent modification of the method, which uses one-sided approximation, does not lead to oscillation of the estimates, but it can only yield a lower bound for the estimates. In this contribution we find an upper bound using the vector-meson dominance model. We show that reasonable constraints for the estimates can be obtained only for large values of the lower limit and relatively small values of the upper limit of the dispersive integral.

1 Introduction

The ACD method is used to evaluate dispersive integrals in QCD (or QCD-like theory) using the first few terms of the Operator Product Expansion (OPE). The ACD method was developed for QCD phenomenology [1] with the help of the OPE for subasymptotic QCD [2]. It uses local quark-hadron duality. The ACD approximates the kernel of a dispersive integral by a polynomial, which enables the integration in the complex plane.

The ACD has been shown to be an ill-posed problem: small variations in the input parameters may lead to large variations in the results [3, 4]. The reliability of the ACD-derived estimates has been investigated for model spectra [5]. It was found the error was difficult to control and the estimates tend to oscillate and even diverge for some models. The oscillatory behavior of the estimates is due at least in part to the nature of the best-L_p approximations and it is more pronounced for greater p [5].

Nasrallah proposed a modified ACD [6], which approximates the $1/s$ kernel by truncated Taylor series around the upper cut-off R. This method has found several

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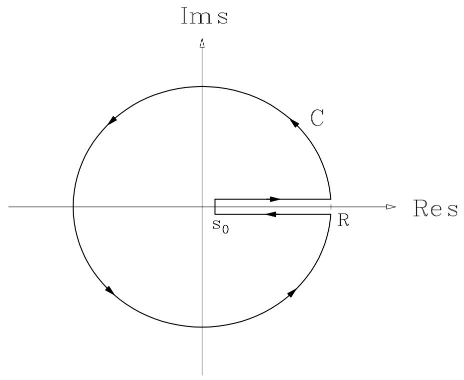

Figure 1: The contour of integration.

applications [7, 8]. We have shown that this method tends to provide only a lower bound for the ACD-derived estimates [4]. The modified method is more stable, but its dependence on the upper cut-off is more pronounced than for the original method. This is a serious problem since R cannot be determined accurately from the physics. Indeed, we can take any R within the region where OPE – or, more generally, a large-momentum expansion – is valid.

Since the modified ACD provides only a lower bound for the estimates the question arises if an upper bound can also be obtained. This is our main objective here. We adopt the simplest model spectrum that is consistent with the QCD dynamics, namely, the vector-meson dominance model.

2 ACD Method

Suppose the function $F(s)$ is analytic in the entire complex plane except on the branch cut along the real axis from s_0. Therefore, the function is analytic within the contour C shown in Fig. 1 and Cauchy's integral formula applies. Eventually we obtain [5]

$$F(0) = \frac{1}{2\pi i} \oint_{|s|=R} \frac{F(s)}{s} ds + \frac{1}{\pi} \int_{s_0}^{R} \frac{\mathrm{Im} F(s)}{s} ds. \tag{1}$$

In contrast to the approach that leads to a dispersion relation, the ACD method makes the *second* integral in (1) vanish. This is achieved by approximating the kernel $1/s$ by a polynomial. The polynomial

$$p_N(s) = \sum_{n=0}^{N} a_n(N) s^n$$

is analytic in the entire complex plane and, therefore, the product $p_N(s)F(s)$ is analytic inside the contour C. Applying Cauchy's theorem we get

$$F(0) = \frac{1}{2\pi i} \oint_{|s|=R} \left[\frac{1}{s} - p_N(s)\right] F(s)\, ds + \frac{1}{\pi} \int_{s_0}^{R} \left[\frac{1}{s} - p_N(s)\right] \mathrm{Im} F(s)\, ds. \tag{2}$$

Eq. (2) is exact since no approximation has been introduced yet. If we find $p_N(s)$ that converges to $1/s$ on the interval $[s_0, R]$ then the second integral in (2) will tend to be small:

$$\overline{F} = \frac{1}{2\pi i} \oint_{|s|=R} \left[\frac{1}{s} - p_N(s)\right] F(s)\, ds + \delta F_{\rm fit}(N) = \overline{F}_N + \delta F_{\rm fit}(N), \qquad (3)$$

where $\delta F_{\rm fit}(N)$ will be called "fit error". The fit error depends not only on N but also on 1) the interval $[s_0, R]$ and 2) the fit routine. The fit routines will be considered later.

Let us assume that the large-s expansion

$$F(s) = \sum_{m=1}^{M} \frac{h_m(s)}{s^m} + O\left(\frac{1}{s^{M+1}}\right) \qquad (4)$$

is valid around the circle $|s| = R$. The expansion (4) may be OPE, which is an asymptotic series, but it may also be the large-momentum expansion of an exactly known vacuum polarization, which may be convergent series. The truncation of the series (4) and neglecting the s-dependence of h_m's intrtoduces two further types of error (see [5, 4] for details). The only non-zero contributions to the first integral in (2) come from the simple poles so that we find

$$F_{\rm ACD} = -\sum_{n=0}^{\min(N,M-1)} \hat{h}_{n+1} a_n(N). \qquad (5)$$

If the spectral function is known exactly, the dispersion relation gives a reliable estimate of $\overline{F}$ so the comparison to $F_{\rm ACD}$ will determine if the ACD estimate is reliable.

2.1 Best-L_p Polynomial Approximations

The usual fit routines minimize the L_p norm

$$d_p = \left[\int_a^b |d(x)|^p w(x)\, dx\right]^{1/p},$$

where

$$d(x) = \frac{1}{x} - p_N(x)$$

and $w(x)$ is the weighting function. The most natural choices are $p = 1, 2, \infty$ and $w(x) = 1$, but there is no *a priori* reason to favour the (unweighted) least-square fit (which was apparently the case in nearly all of the ACD applications thus far).

2.2 Truncated Taylor Series Approximation

An approximation of the function $1/s$ by Taylor expansion around $s = R$ has been proposed [6]. The primary motivation is that such an approximation emphasizes the better-known part of the spectrum, although the strong dependence of the (best-L_p derived) ACD estimates on the lower cut-off may have played a role as well.

For $0 < s \leq R$ the function $1/s$ can be expanded around $s = R$ in geometric series:

$$\frac{1}{s} = \frac{1}{R}\sum_{n=0}^{+\infty}\left(1 - \frac{s}{R}\right)^n . \tag{6}$$

With the help of the binomial formula we get the coefficients of the polynomial $p_N(s)$:

$$a_n(N) = \frac{(-1)^n}{R^{n+1}}\binom{N+1}{n+1}. \tag{7}$$

We have obtained a simple explicit formula for the fit coefficients, which was not possible for the best-L_p approximations (except for very low values of N).

Since $0 < s \leq R$ the difference $d(s)$ is always non-negative:

$$d(s) = \frac{1}{s} - p_N(s) = \frac{1}{s}\left(1 - \frac{s}{R}\right)^{N+1} \geq 0. \tag{8}$$

The approximation has the effect of making $d(s)$ and its derivatives up to the order N vanish at $s = R$. However, the polynomial $p_N(s)$ is not actually an approximation to the function $1/s$ as it only bounds it from below.

3 The lower and upper bound for the ACD estimates

When investigating the stability of the ACD-derived results we introduced two simple model functions $F(s)$ for which the dispersive integral was evaluated exactly. The three types of error in the ACD results were calculated. In the vector-meson dominance model there is one inifinitely sharp (δ-function) resonance in each channel while in the Breit-Wigner model we have one finite-width resonance in each channel. Considerable difference between the models was seen for the original ACD, but for the modified ACD there was – somewhat surprisingly – essentially no difference in the results. Hence we have adopted the much simpler vector-meson dominance model here.

The spectral function of the model is:

$$\text{Im}F(s) = -\pi[f_V^2\delta(s - m_V^2) - f_A^2\delta(s - m_A^2)],$$

$$F(s) = \frac{f_V^2}{s - m_V^2 + i\epsilon} - \frac{f_A^2}{s - m_A^2 + i\epsilon}.$$

Imposing the first and second Weinberg sum rules one obtains [5, 4]

$$\overline{F} = (1 + r)\frac{f^2}{m_V^2}, \tag{9}$$

where $r = m_V^2/m_A^2$. The QCD value is $r \approx 0.4$.

The large-momentum expansion for this model is

$$F(s) = \sum_{n=0}^{\infty} \frac{f_V^2 m_V^{2n} - f_A^2 m_A^{2n}}{s^{n+1}} = \frac{f^2}{s} + \sum_{n=2}^{\infty} \frac{f^2 m_V^{2n}}{s^{n+1}} X_{2n}(r),$$

where

$$X_n(r) \equiv \frac{1 - r^{1-\frac{n}{2}}}{1-r}. \tag{10}$$

The coefficients of the expansion do not depend on s so that for $M > N$ the only error for this model is the fit error, which is always non-negative [4], falls off monotonously with N so $F_{\rm ACD}$ approaches $\overline{F}$ from below. Thus $F_{\rm ACD}$ is only a lower bound for $\overline{F}$. In order to get a close bound we need to decrease R, but obviously not into the region of the resonances.

In general, only a few terms of the OPE are known [1, 2] so in actual applications $M \leq 5$ while for N the logical choice is $N = M - 1$, although other values are possible.

3.1 The upper bound for the estimates

From Eq. (8) we can easily see that the function $d(s)$ has a maximum at $s = s_0$:

$$d(s_0) = \frac{1}{s_0} - p_N(s_0) = \frac{1}{s_0}\left(1 - \frac{s_0}{R}\right)^{N+1} \approx \frac{1}{s_0} - \frac{N+1}{R}, \tag{11}$$

which can be used to obtain a crude upper bound for the estimates:

$$\tilde{p}_N(s) = p_N(s) + d(s_0) \tag{12}$$

so that $\tilde{p}_N(s) \geq 1/s$ for all $s \in [s_0, R]$. Obviously, is a constant and, therefore, Eq. (12) modifies only the coefficient a_0:

$$\tilde{a}_0 = a_0 + d(s_0).$$

Hence instead of the estimate (5) we have

$$\tilde{F}_{\rm ACD} = F_{\rm ACD} + d(s_0) h_1. \tag{13}$$

According to (7)

$$a_0(N) = \frac{N+1}{R}$$

so that the upper bound $\tilde{F}_{\rm ACD}$ will depend on R very weakly. It is clear that the upper bound will depend on s_0 very strongly. Since s_0 is generally small (in the units of the mass of the lowest-lying resonance squared) the upper bound for F will tend to be very large. R needs to be fairly small in order to get a good lower bound (i. e. the claimed central value of [7, 8]). Due to our freedom to choose R within broad limits, the lower bound may approach the exact value, but s_0 is usually constrained by physics; it cannot be raised above the threshold of the process under consideration.

The results are shown in Table 1. We can see that the lower bound is close to the exact value only for small values of R while the upper bound is far above the exact value for this fixed value of s_0.

Table 1: The lower bound F_{ACD} and the upper bound $\tilde{F}_{\text{ACD}}$ in units of the exact value (9) for $M = N + 1$. The lower cut-off is $s_0 = 0.2 m_V^2$ and $r = 0.4$.

M	$R(m_V^2)$	F_{ACD}	$\tilde{F}_{\text{ACD}}$
3	4	0.508	3.570
	5	0.414	3.574
	7	0.301	3.575
	10	0.212	3.574
4	4	0.627	3.536
	5	0.524	3.557
	7	0.390	3.570
	10	0.279	3.573
5	4	0.719	3.483
	5	0.616	3.528
	7	0.470	3.560
	10	0.342	3.570

4 Conclusions

The advantage of the modified ACD method is primarily its stability. However, for realistic values of M the results are far from the expected values even for the simple model spectra. The modified ACD tends to yield a *lower bound* for $\overline{F}$, at least for QCD dynamics.

The results depend very strongly on the upper cutoff R; the estimates may approach the expected value only for very low R (with $M \leq 5$). The upper bound, on the other hand, heavily depends on the lower cut-off, which is usually constrained by the physics. If a low value of R is justifiable and the problem has a particularly high threshold – $s_0 \approx 0.5 m_V^2$ or so – then the modified ACD may provide reasonable constraints for the quantity under investigation.

In many ways, neither the original nor the modified ACD methods are satisfactory. However, the ACD is capable of producing reasonable estimates under certain conditions. Therefore, it would be interesting to investigate further modifications of the method that would yield less oscillatory results that the original ACD while being much more accurate than the modified method.

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